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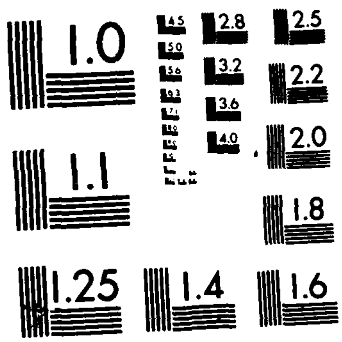

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A MODEL FOR ORDINAL FILTERING OF DIGITAL IMAGES

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A MODEL FOR ORDINAL FILTERING OF DIGITAL IMAGES

By: M. F. JANOWITZ

§1. Introduction. A monochromatic digital image may be thought of as a finite rectangular array of numbers. Each number in the array represents some sort of average value for a specified type of radiation from a fixed geographic or geometric region, with adjacent numbers corresponding to adjacent regions. The digital image is subsequently transformed to an actual picture by means of some sort of video display device. As will soon be seen, however, a given digital image can produce more than one picture, so one must examine the meaning of the display of this type of data. This question will be examined within the context of a model that allows one to classify various filter techniques, and to compare competing techniques so as to decide which is better. In that the underlying model is order theoretic in nature, it is appropriate at this point to present some needed background material from the theory of partially ordered sets. Though the material can be presented in a more abstract fashion, we choose to stay within the image processing framework in which it will be applied.

In what follows, X will denote a finite rectangular array of points, with X having m rows and n columns. The point in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column is denoted $x(i,j)$ or simply x if it is not important to specify the exact coordinates. A monochromatic digital image may then be thought of as a mapping $d : X \longrightarrow L$ where L denotes a finite

chain. Though the exact nature of L may vary from computer to computer, a typical representation may be obtained by taking $L = \{0,1,\ldots,255\}$ where $i \leq j$ in L has its usual meaning in the real number system. For M a nonempty subset of L, we agree to let $\vee M$ and $\wedge M$ denote respectively the largest and smallest elements of M. The symbol P' or $P'(X)$ will be used to denote the collection of nonempty subsets of X, partially ordered by set inclusion. In connection with this, the symbols $\subset$, $\subseteq$, $\cup$, $\cap$ will have their usual set theoretic meanings.

Now let $d : X \longrightarrow L$ be any mapping. Notice that d may be uniquely extended to a join homomorphism $D^*: P'(X) \longrightarrow L$ by the rule

$$D^*(Y) = \vee\{d(y) : y \in Y\}.$$

One can also associate with d a mapping

$$D : L[h_d,255] \longrightarrow P'$$

where

$$h_d = \wedge\{d(x) : x \in X\}$$

by the rule

$$D(h) = \{x \in X : d(x) \leq h\}.$$

The pair (D^*,D) of mappings has a number of interesting features in that

(R1) D^* is isotone

(R2) D is isotone

(R3) $Y \subseteq D[D^*(Y)]$ for all $Y \in P'(X)$

(R4) $h \geq D^*[D(h)]$ for all $h \geq h_d$.

This says that $D^*:P' \longrightarrow [h_d,255]$ is residuated in the sense of [ ], p. 11. The reader is referred to that source for a development of the properties of residuated and residual mappings. What we shall need is the fact that we have established one-one correspondences between three classes of objects:

(A) mappings $d : X \longrightarrow L$

(B) residuated mappings $D^*:P'(X) \longrightarrow [h_d,255]$

where $h_d = \wedge\{d(x) : x \in X\}$

(C) residual mappings $D : [h_d,255] \longrightarrow P'(X)$.

The link from (B) to (A) is accomplished by observing that d is the restriction of D^* to singleton subsets of X with the remaining links arising from the theory of residuated mappings [1].

Suppose now that $\theta : L \longrightarrow L$ is isotone. If $M = [\theta(0),255] = \{h : h \geq \theta(0)\}$, it is easy to see that if θ is regarded as a mapping from L into M, then θ is residuated with $\theta^+ : M \longrightarrow L$ given by

$$\theta^+(k) = \vee\{h : \theta(h) \leq k\}.$$

This may be illustrated by defining θ on $\{0,1,2,3,4,5,6,7\}$ by means of the table

h	0	1	2	3	4	5	6	7
$\theta(h)$	3	3	4	5	6	6	6	6

Then $M = \{3,4,5,6,7\}$ and θ^+ is given by

h	3	4	5	6	7
$\theta^+(h)$	1	2	3	7	7

There is one final needed item. If $d:X \longrightarrow L$ and $\theta:L \longrightarrow L$ is isotone, then $\theta d:X \longrightarrow L$ with $\theta \circ D^*$ its associated residuated mapping. As in the proof of [2], Lemma 7.1, p. 68,

$$(\theta \circ D^*)^+ = D \circ \theta^+.$$

In summary then, a picture may either be viewed in the form (A), (B) or (C) as previously described. Somewhat closer contact may be made with the model described in [2] by working with the lattice of all subsets of X, $P(X)$. In a manner similar to that described earlier in the section, one establishes a bijection between the following classes of objects:

($\bar{A}$) mappings $d:X \longrightarrow L$

($\bar{B}$) residuated mappings $D^*:P(X) \longrightarrow [h_d,255]$ where h_d is some fixed lower bound for $\{d(x) : x \in X\}$

($\bar{C}$) residual mappings $D:[h_d,255] \longrightarrow P(X)$.

The situation will be examined in more detail in the next section.

§2. Display of digital images. Let $d: X \longrightarrow L$ be a monochromatic digital image, where L might be $\{0,1,\dots,255\}$. The digital image d is converted to an actual picture by means of some sort of visual display device. The display device either produces a black and white picture by associating with each integer i in L an intensity of grey $C[i]$, or it produces a color picture by associating with i a color $C[i]$. In the latter situation, the display is controlled by a color lookup table of the form

0	1	2	....	255
C[0]	C[1]	C[2]	....	C[255]

where the 256 colors are chosen from a palette consisting of 2^{3k} possible colors with k representing the number of bits assigned to represent a given color. The colors in the table are then linearly ordered by the rule

$$C[i] < C[j] \quad \text{if} \quad i < j \quad \text{in} \quad L.$$

For purposes of this discussion, it will be assumed that the colors or grey shades $C[0], C[1], \dots, C[255]$ are fixed. Many image processing systems come equipped with a means of dynamically modifying the lookup table so as to enhance certain portions of the picture. This deserves a careful consideration.

We are given a digital picture d where

$$d : X \dashrightarrow L.$$

If θ is an isotone mapping on L, a new picture may be created by looking at

$$\theta d : X \longrightarrow L.$$

But this involves applying θ to each and every pixel $x \in X$ to change the color representation from

$$x \longrightarrow d(x) \longrightarrow C[d(x)]$$

to

$$x \longrightarrow \theta d(x) \longrightarrow C[\theta d(x)].$$

The same effect may be achieved by changing the lookup table so that it becomes

0	1	2	. . .	255
$C[\theta(0)]$	$C[\theta(1)]$	$C[\theta(2)]$	. . .	$C[\theta(255)]$.

Since this only involves changing 256 entries, it is the preferred approach. This, however, has dramatic implications. When a digital image is entered into the computer, one can obtain not just a single picture but a whole class of pictures that arise by means of lookup table modifications. This should cause one to examine the significance of the actual data values in a given picture d.

An alternate view of a digital picture involves associating with each color $C[h]$ that portion of the picture that is painted with the color $C[h]$. Thus one would look at

$$C[h] \longrightarrow h \dashrightarrow \{x \in X : d(x) = h\}$$

instead of

$$x \dashrightarrow d(x) \dashrightarrow C[d(x)] .$$

What happens to the alternate representation when an isotone mapping θ is applied to d? As long as the restriction of θ to the range of d is injective, this is easy to answer, since one simply looks at

$$C[\theta(h)] \longrightarrow h \longrightarrow \{x : d(x) = h\}.$$

What happens though if θ merges distinct levels of the picture? One can certainly still consider

$$C[\theta(h)] \longrightarrow \{x : \theta d(x) = \theta(h)\},$$

but it turns out to be more useful to do a second alternate representation by means of

$$C[h] \longrightarrow \{x : d(x) \leq h\}.$$

Since for $h \geq h_d = \wedge\{d(x) : x \in X\}$ it is true that

$$\theta d(x) \leq h \Longleftrightarrow d(x) \leq \theta^+(h),$$

we then have

$$C[h] \longrightarrow \theta^+(h) \longrightarrow \{x : d(x) \leq \theta^+(h)\} .$$

The second alternate representation associates with each color $C[h]$ those pixels x that are painted with $C[h]$ or some color that preceded it in the lookup table.

§3. <u>Filters</u>. It is often desirable to take a picture $d : X \longrightarrow L$ and enhance it to produce a picture $F(d) : X \longrightarrow L$. This is done by applying some sort of spatial or frequency domain filter to d. To achieve the most generality, let us agree to call any transformation of a picture $d : X \longrightarrow L$ to a picture $F(d) : X \longrightarrow L$ a <u>filter</u>. It is understood, however, that for a filter to be useful, it must somehow or other aid in the understanding of the original picture. At any rate, applying a filter to d produces $F(d)$. This may be illustrated schematically by

$$\boxed{\;d\;} \xrightarrow{\;F\;} \boxed{F(d)}\,.$$

We now apply an isotone mapping θ to the lookup table. This transforms both the original and filtered versions of the picture as indicated below:

$$\begin{tikzcd}
\boxed{d} \arrow[r, "F"] \arrow[d, "\theta"'] & \boxed{F(d)} \arrow[d, "\theta"'] \\
\boxed{\theta d} & \boxed{\theta F(d)}
\end{tikzcd}$$

Suppose now that an object is observed in $\theta F(d)$ that was not observed in any of the other images. Is it real or is it an artifact of F and/or θ? Unless one understands the relation between θd and $\theta F(d)$, this is difficult to answer. An ideal relationship would be if

$$F(\theta d) = \theta F(d),$$

since this says that the process that produced $F(d)$ from d also produced $\theta F(d)$ form $\theta(d)$. This now leads to a natural question. For a given class of isotone mappings $\theta : L \longrightarrow L$, what filters F have the property that

$$F(\theta d) = \theta F(d)$$

for all digital images $d : X \longrightarrow L$? In connection with this, it will often be useful to view a digital image $d : X \longrightarrow L$ as a residual map $D : M \longrightarrow P'(X)$ where $M = [h_d, 255]$ and $h_d = \wedge\{d(x) : x \in X\}$. The filter

$$d \longrightarrow F(d)$$

then takes the form

$$D \longrightarrow \bar{F}(D)$$

where $\bar{F}(D)$ is the residual mapping associated with $F(d)$. Notice then that by [2], Lemma 7.1, p. 60,

$$F(\theta d) = \theta F(d) \Longleftrightarrow \bar{F}(D \circ \theta^{+}) = \bar{F}(D) \circ \theta^{+} .$$

<u>Definition</u>. (i) Let $M \subseteq L$ and $\theta : L \longrightarrow L$ be isotone. The filter F is said to be (θ, M) - <u>compatible</u> in case

$$F(\theta d) = \theta F(d)$$

for all images d with range contained in M.

(ii) The filter F is said to be <u>monotone equivariant</u> if it is (θ, M) compatible for all θ such that $\theta|_M$ is injective.

Our first goal will be an examination of the meaning of monotone equivariance for filters F. It should be mentioned that though this has essentially already been done in [3], it will still be useful to repeat the proofs here.

<u>Lemma</u> 1. If F is monotone equivariant, then range $F(d) \subseteq$ range d, for every $d : X \longrightarrow L$.

<u>Proof</u>: If $h \in$ range d, one can define $\theta : L \longrightarrow L$ so that $\theta(k) = k$ for all $k \in$ range d, but $\theta(h) \neq h$. Then $d = \theta d$, whence

$$F(d) = F(\theta d) = \theta F(d).$$

If $k = (Fd)(x)$, then

$$k = (Fd)(x) = \theta[(Fd)(x)] = \theta(k).$$

In that $h \neq \theta(h)$ we conclude that $h \notin$ range $F(d)$.

<u>Theorem</u> 2. If F is monotone equivariant and if d has range $h_0 < h_1 < \ldots < h_t$, then $F(d)$ depends only upon the sequence

$$D(h_0) \subset D(h_1) \subset \ldots \subset D(h_t)$$

and is independent of the actual levels $h_0 < h_1 < \ldots < h_t$.

<u>Proof</u>. Let $d : X \longrightarrow L$ be as above, and let $Y_i = D(h_i)(i = 0,\ldots 1,\ldots,t)$. In view of Lemma 1, the output is determined by the sequence

$$(\bar{F}D)(h_0) \subseteq (\bar{F}D)(h_1) \subseteq \ldots \subseteq (\bar{F}D)(h_t)$$

since $h_i \leq h < h_{i+1}$ implies $(\bar{F}D)(h) = (\bar{F}D)(h_i)$. To show that the output is independent of the actual levels h_i, choose a sequence

$$k_0 < k_1 < \ldots < k_t$$

of elements of L and define

$$c : X \longrightarrow L$$

so that $Y_i = C(k_i)$. If $\theta : L \longrightarrow L$ is defined so that $\theta(h_i) = k_i$ one then has

$$\theta d = c,$$

whence

$$F(c) = F(\theta d) = \theta F(d).$$

Then

$$\bar{F}(C) = (\bar{F}D) \circ \theta^+$$

and

$$(\bar{F}C)(k_i) = (\bar{F}D)(\theta^+(k_i)) = (\bar{F}D)(h_i).$$

This shows a monotone equivariant filter to be ordinal in nature. It transforms a strictly increasing sequence

$$A_0 \subset A_1 \subset \ldots \subset A_t = X,$$

of subsets of X to an increasing sequence

$$B_0 \subseteq B_1 \subseteq \dots \subseteq B_t = X,$$

where B_i depends on the entire input sequence.

Remark 3. The converse of Theorem 2 is also true. To see this assume that d has range $h_0 < h_1 < \dots < h_t$, that range $F(d) \subseteq$ range d, and that $F(d)$ depends only on the sequence

$$D(h_0) \subset D(h_1) \subset \dots \subset D(h_t) = X$$

of subsets of X. If $\theta|_{\text{range } d}$ is injective, we can let $k_i = \theta(h_i)$ and $c = \theta(d)$. Then

$$d(x) \leq h_i \iff c(x) \leq k_i$$

so that $C(k_i) = D(h_i)$ for $i = 0,1,\dots,t$. But then

$$(\overline{F}C)(k_i) = (\overline{F}D)(h_i),$$

so

$$(Fc)(x) \leq k_i \iff (Fd)(x) \leq h_i$$

and consequently

$$(Fc)(x) = k_i \iff (Fd)(x) = h_i.$$

It is immediate that

$$F(\theta d) = \theta F(d)$$

as desired.

In the most general situation, a change of lookup table involves the application of a linear or piecewise linear mapping θ to L. In the linear case, one is given non-negative integers a, b such that

$$0 \leq a < b \leq 255$$

and $\theta_{a,b}$ is defined by the rule

$$\theta_{a,b}(h) = \begin{cases} 0 & h \leq a \\ [\frac{h - a}{b - a} \cdot 255] & a \leq h \leq b \\ 255 & h > b \end{cases}$$

where $[t]$ denotes the result of rounding the real number t to the nearest integer. When $b = a + 1$, we may denote $\theta_{a,b}$ as θ_a and observe that

$$\theta_a(h) = \begin{cases} 0 & h \leq a \\ 255 & h > a \end{cases} .$$

It turns out that (θ,M) - compatibility for all θ_a is in itself a powerful condition on a filter F.

<u>Definition</u>. A filter F is said to be flat if there is an isotone mapping $\gamma : P(X) \longrightarrow P(X)$ such that

$$\bar{F}(D) = \gamma \circ D$$

for all $d : X \longrightarrow L$.

Theorem 4. Let F be (θ_a, M) - compatible for all θ_a $(0 \le a < 255)$ and all $M \subseteq L$. Then F is flat.

Proof: Let $c, d : X \longrightarrow L$ be digital images, and let $a, b \in L$ be such that $C(a) = D(b)$. We claim that $(\bar{F}C)(a) = (\bar{F}D)(b)$. This is a direct consequence of the fact that

$$(C \circ \theta_a^+)(h) = \begin{cases} C(a) & h \neq 255 \\ X & h = 255 \end{cases}$$

$$(D \circ \theta_b^+)(h) = \begin{cases} D(b) & h \neq 255 \\ X & h = 255. \end{cases}$$

Thus $\theta_a c = \theta_b d$, so

$$[\bar{F}(C)] \circ \theta_a^+ = [\bar{F}(D)] \circ \theta_b^+$$

and

$$[\bar{F}(C)](a) = [\bar{F}(D)](b).$$

We may now define $\gamma : P(X) \longrightarrow P(X)$ by

$$\gamma(Y) = [\bar{F}(D)](b)$$
$$\gamma(\phi) = \phi \quad ,$$

where $D(b) = Y$ and note that γ is well defined. It is then clear that

$$[\bar{F}(D)](h) = (\gamma \circ D)(h)$$

so F is flat.

Corollary 5. If F is (θ_a, M) - compatible for every θ_a and every $M \subseteq L$, then F is (θ, M) - compatible for all isotone mappings $\theta : L \longrightarrow L$ and all $M \subseteq L$.

In view of the above discussion it is reasonable to consider the theory underlying flat filters. Such filters are easy to construct as one only needs to produce an isotone mapping γ on $P(X)$ such that $\gamma(X) = X$, and $\gamma(\phi) = \phi$. Having recognized this need, it is appropriate to decide upon reasonable conditions for these mappings. The idea is to attach some spatial significance as to whether a point belongs to the output of the mapping. In other words, if $Y \subseteq X$, the decision as to whether $x \in \gamma(Y)$ should be based on all points in some small region surrounding x. This may be precisely stated by saying that the mapping γ shall be point-based in that for each $x \in X$ there is a subset $N(x)$ of X containing x and a mapping $\gamma_x : P(N(x)) \longrightarrow \{0,1\}$ such that:

(PB1) $\gamma_x(N(x)) = 1$, and $\gamma_x(\phi) = 0$.

(PB2) For $A \in P(X)$, $x \in \gamma(A) \Longleftrightarrow \gamma_x(A \cap N(x)) = 1$.

So that γ will be independent of the polarity of the image, it turns out to be useful to have γ preserve complements in that

(PB3) For $A \in P(X)$, $\gamma(X \setminus A) = X \setminus \gamma(A)$.

Remark 6. For a point-based isotone mapping γ on $P(X)$ to preserve complements, it is clearly necessary and sufficient that γ_x have the property that for every proper subset A of $N(x)$, exactly one of $\gamma_x(A)$ and $\gamma_x(N(x) \setminus A)$ shall equal 1.

Now let γ be point-based with an associated family of neighborhoods $N(x)(x \in X)$. If $M(x)$ is a second such family, it is evident that so is $M(x) \cap N(x)$. In view of this, there is no harm in assuming

(PB4) If $M(x)(x \in X)$ satisfies (PB1) and (PB2), then $N(x) \subseteq M(x)$.

Such a minimal family of subsets of X will be called the <u>system of neighborhoods associated with</u> γ. Unless otherwise specified, when we speak of a point-based isotone mapping γ, it will be assumed that the family $\{N(x) : x \in X\}$ represents this minimal system of neighborhoods.

In order for γ to interact properly with planar rigid motions, we need to consider the effect of translations. These are mappings of the form $T_{p,q}$ where p,q are fixed integers and $T_{p,q}(i,j) = (p + i, q + j)$. Unless $p = 0 = q$ the domain and range of $T_{p,q}$ will be proper subsets of X. We agree to call the point-based isotone mapping <u>translation - invariant</u> provided it satisfies

(PB5) If $N(x)$ is contained in the domain of the translation T, if $N(x) \neq \{x\}$, and if $y = T(x)$, then $N(y) = T(N(x))$ and $\gamma_y = \gamma_x \circ T^{-1}$.

<u>Remark</u> 7. The reason for the concern about whether $N(x) = \{x\}$ is that around the edge of an image there may not be enough room to form the desired neighborhoods, so that one might simply take $N(x) = \{x\}$.

Definition. Let γ be a point-based, translation-invariant isotone mapping on $P(X)$. To say that γ is frequency-defined is to say that there exists an integer j such that if $N(x) \neq \{x\}$, $\gamma_x(A \cap N(x)) = 1$ if and only if $\#(A \cap N(x)) \geq j$. Here $\#Y$ denotes the cardinality of the set Y.

Theorem 8. Let γ satisfy (PB1) through (PB5). Let x be chosen so that $N(x) \neq \{x\}$ and let $k = \#N(x)$. Let j be the least cardinal number for which $\gamma_x(A) = 1$ for some subset A of $N(x)$ with $j = \#A$. Necessary and sufficient conditions for γ to be frequency-defined are that k be odd and $j = (1 + k)/2$.

Proof: Let γ be frequency-defined. If $\#A \leq k/2$, then $\#(N(x) \setminus A) \geq k/2$, so there is a subset B of $N(x) \setminus A$ with $\#B = \#A$. Since $\gamma_x(B) = 0$, this is a contradiction. If k is even, taking $B \subseteq N(x)$ with cardinality $k/2$ will produce a similar contradiction. Thus k is odd and $\#A \geq k/2$. If now $\#B = (1 + k)/2$, then $\#B > \#(N(x) \setminus B)$. It follows that $\gamma_x(B) = 1$ and consequently that $j = (1 + k)/2$.

Suppose conversely that k is odd and $j = (1 + k)/2$. We must show that $\gamma_x(B) = 1$ for all $B \subseteq N(x)$ for which $\#B \geq j$. Since $\#B \geq j$ implies $\#(N(x) \setminus B) < j$, it follows that $\gamma_x(N(x) \setminus B) = 0$ and consequently $\gamma_x(B) = 1$.

The mapping γ is called a join homomorphism in case $\gamma(S \cup T) = \gamma(S) \cup \gamma(T)$ for all $S, T \in P(X)$. There is a dual notion of meet homomorphism, and to say that γ is a homomorphism is to

say that it is both a join and a meet homomorphism. The next theorem shows that these conditions are extremely powerful, and will be cnly met in trivial circumstances.

<u>Theorem</u> 9. Let γ be a point-based isotone mapping on $P(X)$. Then (1a) $\iff$ (1b), (2a) $\iff$ (2b) and (3a) $\iff$ (3b).

(1a) γ is a join homomorphism.

(1b) For each $x \in X$ there is a subset A_x of $N(x)$ such that $x \in \gamma(S) \iff S \cap A_x \neq \phi$.

(2a) γ is a meet homomorphism.

(2b) For each $x \in X$ there is a subset B_x of $N(x)$ such that $x \in \gamma(S) \iff S \supseteq B_x$.

(3a) γ is a homomorphism.

(3b) For each x there is an element y_x of $N(x)$ such that $x \in \gamma(S) \iff y_x \in S$.

<u>Proof</u>: (1a) $\implies$ (1b) One simply takes C_x to be the union of all subsets of $N(x)$ that are mapped to 0 by γ_x and notes that $\gamma_x(C_x) = 0$. The set $N(x) \setminus C_x$ then serves as A_x.

(1b) $\implies$ (1a) If $x \in \gamma(S \cup T)$, then $(S \cup T) \cap A_x \neq \phi$, so $S \cap A_x \neq \phi$ or $T \cap A_x \neq \phi$. Hence $x \in \gamma(S) \cup \gamma(T)$.

(2a) $\implies$ (2b) One takes B_x to be the intersection of all subsets C of $N(x)$ for which $\gamma(C) = 1$.

(2b) $\implies$ (2a) Is obvious, as is (3b) $\implies$ (3a).

(3a) $\implies$ (3b) If B_x is defined as in the proof of (2a) $\implies$ (2b), then $\gamma_x(B_x) = 1$ but $\gamma_x(C) = 0$ for all $C \subset B_x$. It is immediate that B_x is a singleton set, and (3b) follows.

Remark 10. If γ is translation-invariant, it is tempting to say that if A is in the domain of the translation T, then $\gamma[T(a)] = T[\gamma(a)]$. This is only true if both A and $T(A)$ lie in that portion of X on which $N(x) \neq \{x\}$. To illustrate this, let γ be defined by 3 by 3 neighborhoods by the rule that $\gamma_x(A) = 1$ if $\#A \geq 5$ for $A \in P(N(x))$. Let $X = \{1,2,3,4,5\} \times \{1,2,3,4,5\}$, $x = (2,2)$ and $T = T_{1,1}$. Then if A is the subset indicated by 1's in Figure 2(a), one sees from Figs. 2(c) and 2(d) that $T[\gamma(A)] \neq \gamma[T(A)]$.

1 1 1 0 0	0 0 0 0 0	1 1 1 0 0	0 0 0 0 0
1 1 1 0 0	0 1 1 1 0	1 1 1 0 0	0 0 1 0 0
1 1 1 0 0	0 1 1 1 0	1 1 0 0 0	0 1 1 1 0
0 0 0 0 0	0 1 1 1 0	0 0 0 0 0	0 0 1 0 0
0 0 0 0 0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0
A	$T(A)$	$\gamma(A)$	$\gamma[T(A)]$
Fig. 2(a)	Fig. 2(b)	Fig. 2(c)	Fig. 2(d)

§4. Underlying Statistical Considerations. Suppose that the true input data $d^*: X \longrightarrow L$ has been corrupted by additive or multiplicative noise so as to produce an actual input $d : X \longrightarrow L$. One wants to construct a flat filter F that somehow smoothes the noise without blurring the picture. Thus at each level h, one wishes to estimate

$$A_h^* = \{x : d^*(x) \le h\}$$

by means of looking at

$$A_h = \{x : d(x) \le h\}.$$

Hopefully, $F(d)$ will provide a good estimate of A_h by way of

$$\{x : (Fd)(x) \le h\}.$$

To put the problem in a specific framework, suppose that A_h and A_h^* are related by the assumption that for some fixed a priori probability p $(p > 0.5)$ one has

(S1) $x \in A_h^* \Longrightarrow x \in A_h$ with probability p,

(S2) $x \notin A_h^* \Longrightarrow x \notin A_h$ with probability p,

and that these probabilities are independent of the location of x. The goal is to define γ on $P(X)$ so that

(1) γ is isotone

(2) γ somehow maximizes the probability of correct classification of points. In other words, in some specific sense we wish to maximize the a posterior probability that

(a) If $x \in \gamma(A_h)$ then $x \in A_h^*$, and

(b) If $x \notin \gamma(A_h)$ then $x \notin A_h^*$.

In view of the discussion of §3, it seems reasonable to restrict our attention to translation-invariant, point-based isotone mappings having a fixed system $\{N(x) : x \in X\}$ of neighborhoods of points Because of this, we need only consider a fixed point x for which $N(x) \neq \{x\}$, and the mapping $\gamma_x : P(N(x)) \longrightarrow \{0,1\}$ defined by

$$\gamma_x(A) = 1 \iff x \in \gamma(A)$$

for $A \subseteq N(x)$.

For a given subset A_h^* of X, one can define a probability measure p' on $N(x)$ by the rule that for $A \subseteq N(x)$, $p'(A)$ shall be the probability of observing A, given the true input data $A_h^* \cap N(x)$. For example, if $N(x)$ is a 3 by 3 neighborhood centered on x, if $A_h^* \cap N(x)$ and A are given by the matrices

$$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$

where 1 denotes membership in a given set and 0 membership in its complement, we have $p'(A) = p^5(1 - p)^4$. A numerical measure of the ability of γ_x to distinguish between A_h^* and its complement is provided by the <u>gain</u> $G(\gamma_x)$. For $x \in A_h^*$, this is defined to be the probability that for a randomly chosen subset B of $N(x)$, $\gamma_x(B) = 1$;

added to the probability that if the true input data were the complement of A_h^*, we would have $\gamma_x(B) = 0$ for B a randomly chosen subset of $N(x)$. In connection with this it is convenient to let

$$T_\gamma = \{B : B \subseteq N(x),\ \gamma_x(B) = 1\}$$

$$F_\gamma = \{B : B \subseteq N(x),\ \gamma_x(B) = 0\} .$$

It follows that

$$G(\gamma_x) = \Sigma\{p(A) : A \in T_\gamma\}$$
$$+ \Sigma\{p(N(x)\setminus A) : A \in F_\gamma\}.$$

The reason for this is that if p'' is the probability measure associated with the complement of A_h^*, then for $B \subseteq N(x)$,

$$p''(B) = p'(N(x) \setminus B).$$

One may now search for those mappings γ such that γ_x has maximal gain for a given type of input data. The search is aided by the following result:

Theorem 11. Let p' be a probability measure defined on $N(x)$. Suppose p' has the property that

(1) for $A \subseteq N(x)$, $p'(A) \neq p'(N(x)\setminus A)$, and

(2) if $p'(A) > p'(N(x)\setminus A)$ and if $A \subset B \subseteq N(x)$, then $p'(B) > p'(N(x)\setminus B)$.

Define $\gamma_x' : P(N(x)) \longrightarrow \{0,1\}$ by the rule

$$\gamma_x'(A) = \begin{cases} 1 & p'(A) > p'(N(x) \setminus A) \\ 0 & \text{otherwise} \end{cases} .$$

Then γ_x' is isotone; furthermore, $G(\gamma_x') \geq G(\gamma_x'')$ for any other isotone mapping $\gamma_x'' \longrightarrow \{0,1\}$.

<u>Proof</u>: The fact that γ_x' is isotone comes from condition (2) of the theorem. The assertion regarding $G(\gamma_x') \geq G(\gamma_x'')$ follows from an examination of the sets $T_{\gamma'}$, $F_{\gamma'}$, $T_{\gamma''}$ and $F_{\gamma''}$. Any changes in the sums for $G(\gamma_x')$ and $G(\gamma_x'')$ must clearly come from members of

$$(T_{\gamma'} \cap F_{\gamma''}) \cup (T_{\gamma''} \cap F_{\gamma'}).$$

For $A \in T_{\gamma'} \cap F_{\gamma''}$, the term in $G(\gamma_{x'})$ corresponding to A is $p'(A)$, while in $G(\gamma'')$ it is $p'(N(x) \setminus A)$. By construction of γ', we have $p'(A) > p'(N(x) \setminus A)$. Similarly, for $B \in T_{\gamma''} \cap F_{\gamma'}$, the term in $G(\gamma')$ is $p'(N(x) \setminus B)$ while the one in $G(\gamma'')$ must be $p'(B)$. Again by construction of γ', $p'(N(x) \setminus B) > p'(B)$.

<u>Corollary</u> 12. If $N(x) \subseteq A_h^*$ and $k = \#N(x)$ is odd, then the mapping γ_x' described in the theorem is given by

$$\gamma_x(A) = 1 \text{ if } k/2 < \#A.$$

<u>Remark</u> 13. The mapping in Corollary 12 may be implemented by means of applying a suitable median filter to d. This says that within the framework of the current model, the best way of smoothing noise in the interior is to use a median filter.

The paper will conclude by carefully examining the situation where $N(x)$ is a 3 by 3 window centered on x. The members of A_h^* will be denoted by 1's, those of its complement by 0's, and we shall restrict our attention to a fixed x. Notice that the mapping produced by Theorem 11 is necessarily complement preserving. This is desirable because one wants the output of a filter to be independent of the polarity of the input data. In view of this, we shall concentrate on input data for which $x \in A_h^*$ and assume that $5 \leq \#(N(x) \cap A_h^*)$. We shall begin by looking at 5 types of input data. They will be denoted type i $(i = 0,1,2,3,4)$, according to whether or not $N(x)$ has $i = 0,1,2,3,$ or 4 members of $X \setminus A_h^*$. Before doing the theoretical analysis, let us examine some calculated probabilities of correct classifications. This is done in Table 3. There are two items of interest here. For the type 4 case, the filters all seem to decrease the probability of correct classification. This is hardly surprising since in this instance the 8 immediate neighbors of the given pixel all have equal chances of representing members of A_h^* as opposed to its complement. Thus no additional information can be obtained by looking at the 8 neighbors of x. The second observation is that in the type 3 case the answer seems to depend on the numerical value p of the a priori probability of correct classification.

Prob. Correct	Method	TYPE 0	1	2	3	4
.9	A	.9991	.9954	.9787	.9128	.7071
	B	.9962	.9861	.9590	.9112	.8717
	C	.9914	.9515	.9104	.9014	.8980
.8	A	.9804	.9529	.8944	.7839	.6079
	B	.9584	.9233	.8736	.8150	.7487
	C	.9006	.8564	.8212	.8046	.7929
.7	A	.9012	.8467	.7715	.6747	.5606
	B	.8576	.8125	.7629	.7100	.6519
	C	.7757	.7450	.7222	.7057	.6906
.6	A	.7334	.6870	.6367	.5833	.5280
	B	.6963	.6661	.6354	.6043	.5722
	C	.6374	.6250	.6139	.6036	.5936

Method A. $x \in \gamma_A(A_h)$ if $5 \leq \#A_h \cap N(x)$.

Method B. $x \in \gamma_B(A_h)$ if $6 \leq \#A_h \cap N(x)$ or if $\#A_h \cap N(x) \in \{4,5\}$ and $x \in A_h^*$

Method C. $x \in \gamma_C(A_h)$ if $7 \leq \#A_h \cap N(x)$ or if $\#(A \cap N(x)) \in \{3,4,5,6\}$ and $x \in A_h^*$

Table 3. Probabilities of correct classification.

One can bring Theorem 11 to bear on the problem by simply calculating probabilities of occurrence of each of the 2^9 possible subsets of $N(x)$, given the *a priori* assumption regarding the nature of $A_h^* \cap N(x)$. This has been done but the rather tedious details will not be repeated here. Rather, we shall simply list the results in Table 4.

DATA TYPE	Best Isotone Method
0	γ_A of Table 3
1	γ_A of Table 3
2	γ_A of Table 3
3	There is a critical value p_0 of $p(.8958 \le p_0 < .9000)$ such that for $p \ge p_0$, γ_A is the best method and for $p < p_0$, γ_B of Table 3 is best. p_0 is determined by solving the equation $p = x(1 - p)$ where x is the root of the equation $x^6 - 9x^5 + 6x^4 - 24x^3 + 6x^2 - 4x + 1 = 0$ lying between $x = 8.6$ and $x = 9$.
4	$x \to \gamma(A_h)$ if $x \in A_h$.

Table 4. Summary of "best" flat filters based on 3 by 3 neighborhoods.

Up to this point it has been assumed that the center pixel x was a member of A_h^*, and that any other members of $N(x)$ A_h^* were randomly scattered through $N(x) \setminus \{x\}$. The situation changes dramatically if more information is known. To see this, note that $N(x)$ consists of three data types as indicated below:

c	b	c
b	a	b
c	b	c

The center pixel is type a, its North, South, East, West neighbors are type b, and the remaining four corner pixels are type c. Our basic assumption will now be that $x \in A_h^*$, and that we know the number of type b as well as the number of type c pixels that are in the complement of A_h^*, but not their precise location. Thus a data type may be specified by an ordered pair of integers that denotes the number of pixels of types b and c that lie in the complement of A_h^*. For example, data type $(1,2)$ indicates the presence of one type b pixel and two type c pixels in the complement of A_h^*. The calculations inovlved in applying Theorem 11 are again tedious, so they will be omitted. By way of illustration, however, the type $(0,1)$ - case will be considered in some detail.

Recall that p denotes the *a priori* probability of the input data being correct, and set $q = 1 - p$. The probability of observing variour numbers of 1's in the corners is then seen to be

#1's	Probabilities
4	p^3q
3	$p^4 + 3p^2q^2$
2	$3p^3q + 3pq^3$
1	$3p^2q^2 + q^4$
0	pq^3

Since by assumption the type a and b pixels all lie in A_h^*, they can be lumped together into a single category. There are now $6 \times 5 = 30$ data types to analyze. This involves examining 15 pairs of subsets. But this can be cut down to 6 pairs if one realizes two things:

1. If there are more 1's than 0's in both data types, the result is clear.
2. If there are two 1's in the type c data, then the decision rests entirely on the type a and b.

DATA TYPE			
A	$N(x) \setminus A$	$p'(A)$	$p'(N(x) \setminus A)$
(5,0)	(0,4)	• p^6q^3	p^3q^6
(4,0	(1,4)	• p^5q^4	p^4q^5
(3,0)	(2,4)	p^4q^5	• p^5q^4
(5,1)	(0,3)	• $p^5q^4 + 3p^7q^2$	$p^4q^5 + 3p^2q^7$
(4,1)	(1,3)	• $p^4q^5 + 3p^6q^3$	$p^5q^4 + 3p^3q^6$
(3,1)	(2,3)	$3p^5q^4 + p^3q^6$	• $3p^4q^5 + p^6q^3$

NOTE: The dots indicate the higher of the two probabilities, in each row.

The actual results are summarized in Table 5.

DATA TYPE	BEST ISOTONE METHOD $\psi_x(A) = 1$ if
(0,1)	$6 \leq \#N(x) \cap A$ or $5 = \#N(x) \cap A$ unless $N(x) \cap A$ is type (1,4) or $4 = \#N(x) \cap A$ provided $N(x) \cap A$ is type (4,0)
(0,2)	$A \cap N(x)$ has at least 3 members that are type a or b
(1,2)	Same as (0,2)
(1,1)	$6 \leq \#N(x) \cap A$ or $5 = \#N(x) \cap A$ unless $N(x) \cap A$ is type (0,1,4) or (0,4,1) $4 = \#N(x) \cap A$ provided $N(x) \cap A$ is of type (1,3,0) or (1,0,3).

NOTE: The types (1,0), (2,0) and (2,1) are handled by symmetry

Table 5. Summary of "best" flat filters based on 3 by 3 neighborhoods.

As a final item, let us see how well the "best" flat filters do w th various types of data.

DATA	A PRIORI PROBABILITY OF CORRECT DATA			
TYPE	.9	.8	.7	.6
(0,0)	.9991	.9804	.9012	.7334
(0,1)	.9957	.9545	.8498	.6886
(0,2)	.9914	.9421	.8369	.6826
(0,1)	.9802	.8979	.7715	.6371
(1,2)	.9526	.8499	.7311	.6134

Table 6. A posterieri probability of correct classification for data of given type using "best" filter method as specified in Theorem 11.

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Spatial filtering techniques for monochromatic digital images are examined within the context of an ordinal model that is also suitable for cluster analysis. For techniques based on information received from a 3 by 3 window, those filters are specifically described that yield the highest probability of providing correct classification of pixels.

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